

Reconstruction from nonuniform data using the energy reduction,  
the steepest descent and the contraction mapping method

Yongwan Park

Department of Electrical Engineering  
California Institute of Technology  
Pasadena, CA, 91125

**ABSTRACT**

We introduce three iterative methods for reconstructing a band-limited function from its unevenly spaced sampled data. Each iterative method introduces its own varying coefficient which is adaptively determined at each iteration. A varying coefficient for the first algorithm is adaptively determined based on the error energy reduction of the signal. The second method, the method of steepest descent, determines a varying coefficient based on the error energy reduction of the unevenly spaced sampled data. Third, the contraction mapping method determines a varying coefficient based on the distance reduction of the estimated signals. The reconstructed signal from these algorithms converges to the desired signal if the basis functions,  $\{\exp(j\omega t_n)\}$ , where  $\{t_n\}$  are the unevenly spaced sampling points, form a complete set in the signal subspace of the original signal.

**1. INTRODUCTION**

In this paper, we consider the problem of reconstructing a band-limited function from its unevenly spaced sampled data using the iterative methods. Reconstruction from the finite unevenly spaced sampled data has a variety of applications in many areas[5],[7],[8]. Wiley[3] proposed an iterative method to reconstruct a signal from pulse typed nonuniform samples based on the contraction mapping theorem of Sandberg[2]. Marvasti et al[1] applied Sandberg's iterative method to recover a signal from the impulse typed unevenly spaced samples based on Wiley's conjecture. They also showed the convergence of the iteration and determined the range of the coefficient. However, they could not determine the proper choice of the coefficient of the iterative sequence theoretically. Yeh et al[7] applied the method of projections onto convex sets to reconstruct images from nonuniform samples and compared with iterative algorithms of Sauer et al[8] based on the [3].

The energy reduction and the steepest descent methods determine the exact varying coefficient at each iteration based on their own constraints. The error energy reduction and the convergence of each algorithm are proven in this paper. It is also shown that the proposed algorithms with exact varying coefficient give fast convergence to the desired signal compared with the algorithm with a constant coefficient[1]. The contraction mapping also determines a varying coefficient based on its own constraint. We observe the distance reduction of the estimated signals and the convergence to the fixed values.

## 2. THE SEQUENCE OF ITERATION

Let  $f(t)$  be the periodic band-limited signal and  $\{t_n; n = 1, 2, \dots, N\}$  be its unevenly spaced sampled points. We define  $T$  as the nonuniform sampling operator, and  $Q$  as the band-limiting operator.

We introduce the following iterative sequence with a varying coefficient,  $c_i + 1$ , to reconstruct the band-limited function, i.e.,

$$\begin{aligned} f_{i+1}(t) &= f_i(t) + c_{i+1}r_i(t) \\ &= P\{f_i(t)\} \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} r_i(t) &= Q \langle f(t_n) - Tf_i(t) \rangle \\ &= QT e_i(t). \end{aligned} \quad (2.1a)$$

Note that  $e_i(t)$  is the error between the original signal  $f(t)$  and its estimated signal after  $i$ -th iteration,  $f_i(t)$ , i.e.,

$$e_i(t) = f(t) - f_i(t). \quad (2.2)$$

The varying coefficient  $c_{i+1}$  in eq(2.1) can be determined based on the following three algorithms.

## 3. RECONSTRUCTION ALGORITHM

### 3.1. The Error Energy Reduction Algorithm

We first introduce the error energy reduction algorithm. The error energy reduction constraint decreases monotonically the squared norm error (the error energy) between the original and the estimated signal, i.e.,

$$\|f(t) - f_{i+1}(t)\|^2 \leq \|f(t) - f_i(t)\|^2. \quad (3.1)$$

With eq(2.1), the left hand side of the inequality in eq(3.1) can be rewritten as

$$\begin{aligned} \eta_{i+1}(t) &= \|e_i(t) - c_{i+1}r_i(t)\|^2 \\ &= \eta_i(t) - 2c_{i+1} \langle e_i(t), r_i(t) \rangle + c_{i+1}^2 \|r_i(t)\|^2, \end{aligned} \quad (3.2)$$

where  $\eta_i(t) = \|e_i(t)\|^2$  and  $\langle \cdot, \cdot \rangle$  denotes the inner product. We provide the procedure to obtain the exact varying coefficient that minimizes the error energy at each iteration. That is

$$\frac{d}{dc_{i+1}} \eta_{i+1}(t) = 0. \quad (3.3)$$

By substituting eq(3.2) in eq(3.3), eq(3.3) can be rewritten as follows:

$$\begin{aligned}\frac{d}{dc_{i+1}}\eta_{i+1}(t) &= \frac{d}{dc_{i+1}}[\eta_i(t) - 2c_{i+1} \langle e_i(t), r_i(t) \rangle + c_{i+1}^2 \|r_i(t)\|^2] \\ &= -2 \langle e_i(t), r_i(t) \rangle + 2c_{i+1} \|r_i(t)\|^2 \\ &= 0.\end{aligned}\quad (3.4)$$

From eq(3.4),  $c_{i+1}$  is determined to be

$$c_{i+1} = \frac{\langle e_i(t), r_i(t) \rangle}{\|r_i(t)\|^2}. \quad (3.5)$$

### • The error energy reduction and convergence of the algorithm

It is easily shown that the error energy reduction algorithm with a varying coefficient decreases monotonically the error energy of the signal at each iteration. Note that the numerator and the denominator of eq(3.5) can be expressed as follows:

$$\begin{aligned}\langle e_i(t), r_i(t) \rangle &= \sum_{n=1}^N e_i^2(t_n) \\ &= Y_i\end{aligned}\quad (3.5a)$$

and

$$\begin{aligned}\|Tr_i(t)\|^2 &= \|Qe_i(t_n)\|^2 \\ &= Z_i.\end{aligned}\quad (3.5b)$$

With substituting eq(3.5) into eq(3.2), eq(3.2) can be rewritten as follows:

$$\eta_{i+1}(t) = \eta_i(t) - \xi_i \quad (3.6)$$

where  $\xi_i = \frac{Y_i^2}{Z_i}$ . Since  $\xi_i > 0$ , the error energy of the signal can be written as follows:

$$\eta_{i+1} \leq \eta_i \quad \text{for all } i \quad (3.7)$$

This completes the monotonic reduction of the error energy.

We will show that the estimated nonuniform sampling values  $\{f_i(t_n); n = 1, 2, \dots, N\}$  converge to the available nonuniform sampling values  $\{f(t_n); n = 1, 2, \dots, N\}$ , and the estimated signal  $f_i(t)$  converges to the desired signal  $f(t)$ . We write eq(3.6) in a recursive manner as follows:

$$\begin{aligned}\eta_{i+1}(t) &= \eta_i(t) - \xi_i \\ &= \eta_{i-1}(t) - \xi_{i-1} - \xi_i \\ &\vdots \\ &= \eta_0(t) - \sum_{k=1}^i \xi_k.\end{aligned}\quad (3.8)$$

Since  $0 < \eta_{i+1}(t) < \infty$  and  $\xi_i > 0$ , it is true that

$$\sum_{k=1}^{\infty} \xi_k < \infty. \quad (3.9)$$

Therefore,  $\lim_{i \rightarrow \infty} \xi_i$  goes to zero, consequently  $\lim_{i \rightarrow \infty} Y_i^2 \rightarrow 0$  and  $\lim_{i \rightarrow \infty} r_i(t) \rightarrow 0$ . Thus, from eq(2.1a), we see that the estimated nonuniform samples  $\{f_i(t_n)\}$  converge to  $\{f(t_n)\}$ , i.e.,

$$\lim_{i \rightarrow \infty} f_i(t_n) = f(t_n). \quad (3.10)$$

This completes the convergence of nonuniform data.

When the sampling set  $\{t_n\}$  is a sequence such that the non-harmonic Fourier functions  $\{\exp(i\omega t_n)\}$  form a complete basis vectors, then the error of the signal can be written as follows[4]:

$$e_i(t) = \sum_{n=1}^N e_i(t_n) \Psi_n(t) \quad \text{for any } i. \quad (3.11)$$

Note that the interpolating kernel  $\Psi(t)$  is the Lagrange function. From eq(3.10) we can see that  $e_i(t_n)$  goes to zero as  $i \rightarrow \infty$ . Therefore

$$\begin{aligned} \lim_{i \rightarrow \infty} e_i(t) &= \sum_{n=1}^N 0 \cdot \Psi_n(t) \\ &= 0. \end{aligned} \quad (3.12)$$

Since  $e_i(t)$  is the error between  $f(t)$  and  $f_i(t)$ ,  $f_i(t)$  converges to  $f(t)$ , i.e.,

$$\lim_{i \rightarrow \infty} f_i(t) = f(t). \quad (3.13)$$

### 3.2. The Method of Steepest Descent

The method of steepest descent is used to reconstruct a band-limited function from its unevenly spaced sampled data. This method is based on reducing a defined cost function  $J$ . We define the cost function  $J[f(t)]$  as the quadratic form

$$J[f(t)] = \frac{1}{2} \|Tf(t) - f(t_n)\|^2. \quad (3.14)$$

Since the defined cost function  $J[f(t)]$  is the error energy of the unevenly spaced sampled data, this method minimizes the error energy of the nonuniform sampling values at each iteration. The iteration sequence of the method of steepest descent is given by[6]

$$\begin{aligned} f_{i+1}(t) &= f_i(t) + c_{i+1} \nabla J[f_i(t)] \\ &= f_i(t) + c_{i+1} r_i(t). \end{aligned} \quad (3.15)$$

We want to find a varying coefficient which minimizes the cost function. A varying coefficient  $c_{i+1}$  is determined via

$$\frac{d}{dc_{i+1}} J[f_{i+1}(t)] = 0. \quad (3.16)$$

With the substitution of eq(3.15) into eq(3.16), eq(3.16) can be rewritten as

$$\begin{aligned} & \frac{d}{dc_{i+1}} J[f_{i+1}(t)] \\ &= \frac{1}{2} \frac{d}{dc_{i+1}} [J[f_i(t)] - 2c_{i+1} \langle Tf_i(t) - f(t_n), Tr_i(t) \rangle + c_{i+1}^2 \|Tr_i(t)\|^2] \\ &= \langle r_i(t), r_i(t) \rangle + c_{i+1} \|Tr_i(t)\|^2 \\ &= 0. \end{aligned} \quad (3.17)$$

From eq(3.17),  $c_{i+1}$  is found to be

$$c_{i+1} = \frac{\langle r_i(t), r_i(t) \rangle}{\|Tr_i(t)\|^2}. \quad (3.18)$$

#### • The error energy reduction and convergence of the algorithm

In this part it is shown that the method of steepest descent decreases monotonically the error energy of the nonuniform samples and the signal. By substituting eq(3.18) into eq(3.14), the error energy of the nonuniform samples can be written as follows:

$$\begin{aligned} J[f_{i+1}(t)] &= \frac{1}{2} \|Tf_i(t) - c_{i+1}Tr_i(t) - f(t_n)\|^2 \\ &= J[f_i(t)] - \frac{1}{2} \frac{\|r_i(t)\|^2}{\|Tr_i(t)\|^2}. \end{aligned} \quad (3.19)$$

Since  $J[f_{i+1}(t)]$  and the second term on the right side of eq(3.19) are positive, it is true that

$$J[f_{i+1}(t)] \leq J[f_i(t)] \quad \text{for all } i. \quad (3.20)$$

We can also show that the error energy of the signal decreases monotonically by reducing the error energy of the nonuniform sampling data, i.e.,

$$\begin{aligned} \eta_{i+1}(t) &= \eta_i(t) - 2c_{i+1} \langle e_i(t), r_i(t) \rangle + c_{i+1}^2 \langle r_i(t), r_i(t) \rangle \\ &\leq \eta_i(t) - c_{i+1} \langle e_i(t), r_i(t) \rangle. \end{aligned} \quad (3.21)$$

Since  $c_{i+1}$  and  $\langle e_i(t), r_i(t) \rangle$  are positive, it is true that

$$\eta_{i+1}(t) \leq \eta_i(t) \quad \text{for all } i. \quad (3.22)$$

Convergence to the nonuniform samples,  $f(t_n)$ , and the desired signal,  $f(t)$ , can be proved via the same procedure used in the case of the error energy reduction algorithm.

### 3.3. The Contraction Mapping Method

In this part we also consider the process of determining a varying coefficient  $c_{i+1}$  based on the contraction mapping. The distance between the two signals is described as follows:

$$\|Pf_i(t) - Pf_{i-1}(t)\| \leq \rho \|f_i(t) - f_{i-1}(t)\| \quad \text{for all } i. \quad (3.23)$$

If  $0 \leq \rho < 1$ , the operator  $P$  is said to be a contraction mapping. The operator is said to be nonexpansive, when  $\rho = 1$ . If  $\rho = 1$  and eq(3.23) holds with equality only if  $f_i(t) = f_{i-1}(t)$ , then the operator  $P$  is said to be strictly nonexpansive. Since the norm can be interpreted as the distance between two signals, the contraction mapping operation has the property that the distance between two signals tends to decrease after each iteration. Using equation (2.1) in eq(3.23) results in

$$\|c_{i+1}r_i(t)\| \leq \rho \|c_i r_{i-1}(t)\|. \quad (3.24)$$

Thus a varying coefficient  $c_{i+1}$  is found to be expressed as follows:

$$c_{i+1} \leq \rho c_i \frac{\|r_{i-1}(t)\|}{\|r_i(t)\|} \quad (3.25)$$

## IV. EXPERIMENTAL RESULTS

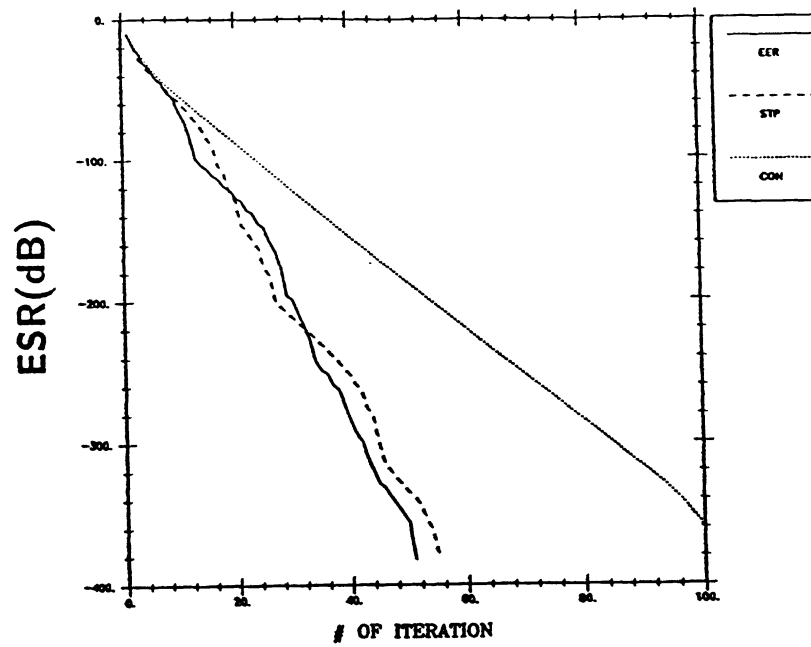
We used the original test signal as the periodic band-limited signal with the band  $B=16$ . The original signal is

$$f(t) = s(t) + 1.3s(3t - 1.3) + 1.1s(10t - 1.1) - .9s(13t + .7) - .6s(16t + .5) \quad (4.1)$$

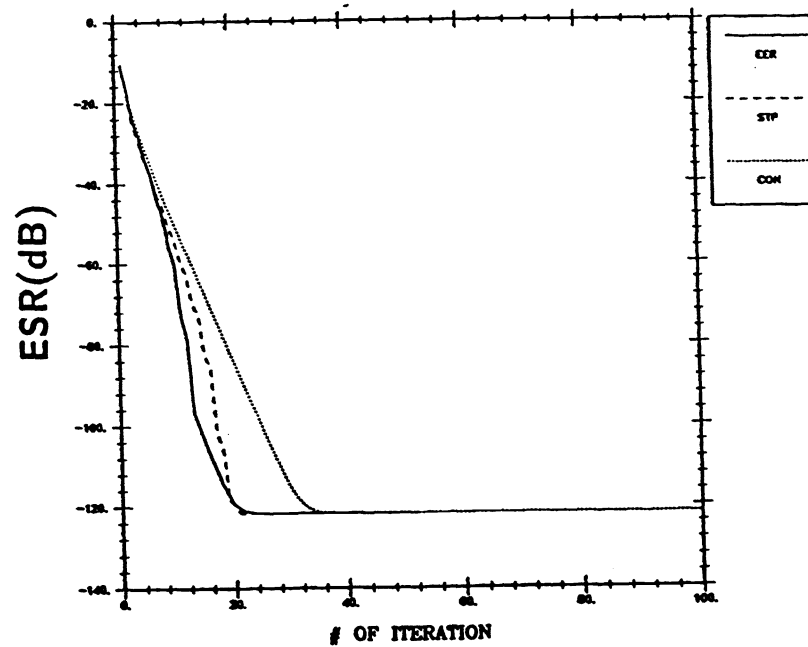
where  $s(t) = \cos(2\pi t)$ . The 33 points were drawn from the random number generator in IMSL/LIB (VERSION 1.1) having uniform distribution in the interval  $[-0.5, 0.5]$ . The subroutine RNUN with ISEED=123457 was used to generate random points. We rescaled the random points to be in the range  $[-1/2B, 1/2B]$ . The unevenly spaced sampled points were obtained by the degrees of the jitter of the rescaled random points from the evenly spaced sampled points.

We reconstructed the signal,  $f(t)$ , using the error energy reduction method(EER), the method of steepest descent(STP) with a varying coefficient. We also implemented the iterative method with a constant coefficient(CON). Figs. 1, 2 and 3 depict the error to signal power ratio(ESR) of the estimator for the signal  $f(t)$  and the nonuniform samples  $f(t_n)$  due to the number of iterations according to the degrees of jitter, i.e., 25%, 50%, and 75% of the Nyquist rate. The ESR is defined as

$$\text{ESR} = 10 \log \frac{|f(t) - f_i(t)|^2}{|f(t)|^2} \quad (\text{dB}). \quad (4.2)$$

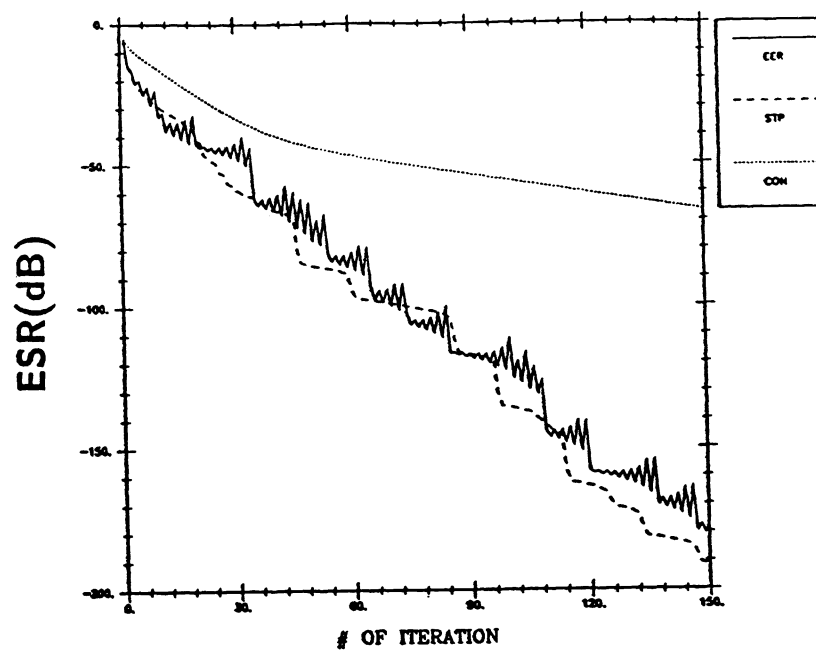


(a)

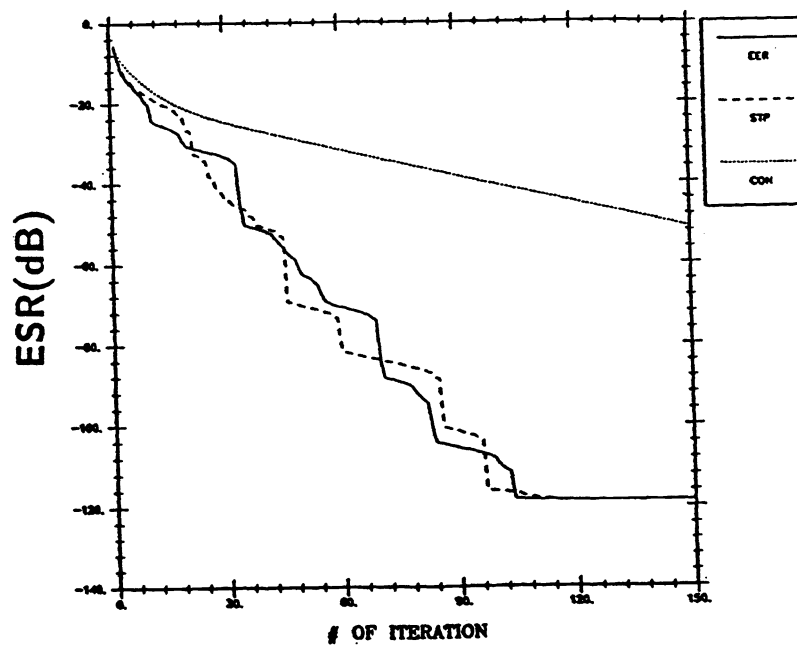


(b)

Figure 1 : The ESR of the estimator (a) for the nonuniform data and (b) for the signal, in dB when  $|t_n - n\Delta| < \Delta/4$ .



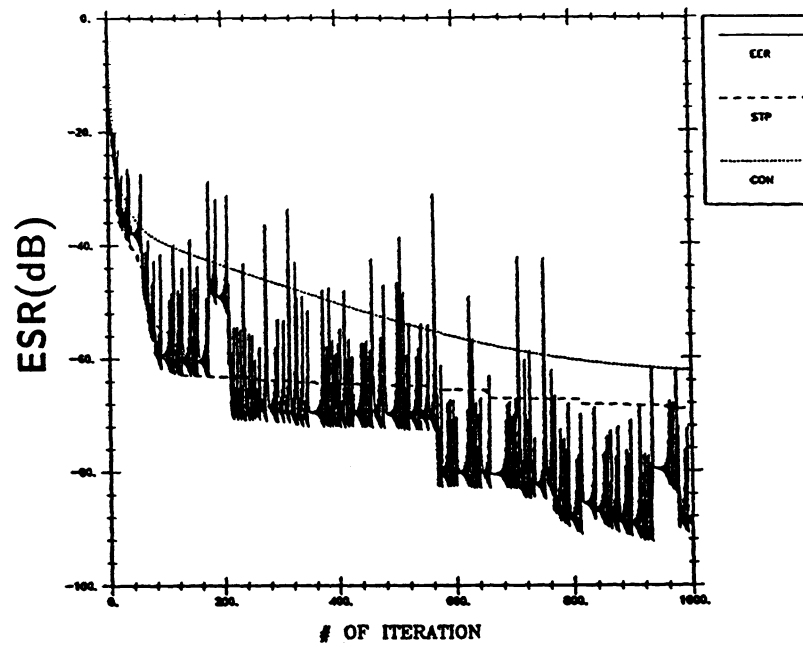
(a)



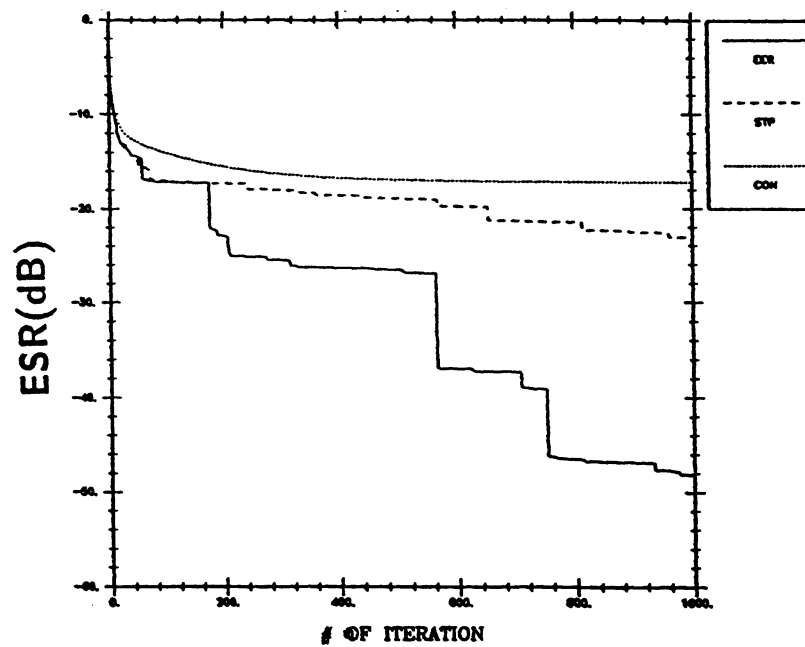
(b)

Figure 2 : The ESR of the estimator (a) for the nonuniform data and (b) for the signal, in dB when  $|t_n - n\Delta| < \Delta/2$ .





(a)



(b)

Figure 3 : The ESR of the estimator (a) for the nonuniform data and (b) for the signal, in dB when  $|t_n - n\Delta| < 3\Delta/4$ .

As expected the error energy reduction method decreased monotonically the error energy of the signal at each iteration. However the error energy of the nonuniform data was not reduced monotonically but finally the estimated nonuniform sampling data converged to the desired nonuniform data. The method of steepest descent decreased monotonically not only the error energy of the nonuniform data but also that of the signal at each iteration.

The methods with a varying and a constant coefficient were used to reconstruct a signal for the 5 different data sets. Table 1 and 2 give the experimental results when the evenly spaced sampled points are jittered 25%, 50% of the Nyquist rate, respectively. We observed the error energy of the first iteration. The EER algorithm was stopped when the ESR is less than -120dB or -100dB based on the degrees of the jitter and compared with other methods.

We also reconstructed  $f(t)$  with a varying coefficient which is determined by the contraction mapping method. Fig.4. shows the distance between  $f_{i+1}(t)$  and  $f_i(t)$ . As expected, the distance decreased monotonically at each iteration in fig.4.a. The error energy of the signal didnot decrease monotonically, however  $f_i(t)$  does finally converge to the fixed values as shown in fig.4.b.

## V. CONCLUSION

We presented three iterative methods to reconstruct a band-limited signal from its unevenly spaced sampled data. Each iterative method introduces its own varying coefficient which is adaptively determined at each iteration. We determined a varying coefficient which minimizes the error energy of signal by maximizing the reduction of the error energy in the first algorithm. A varying coefficient which minimizes the cost function( error energy of the nonuniform data) was found under the steepest descent constraint. We proved not only the error energy reduction of the signal and the nonuniform data but also the convergence of the algorithms. A varying coefficient of the contraction mapping method was determined by the distance reduction of the estimated signals. The proposed iterative algorithms provide stable reconstruction after a few iterations compared with the iterative method using a constant coefficient as shown in tables.

## REFERENCES

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DATA SET	# OF ITERATION	1ST ITERATION (ESR)dB	VARYING COEFFICIENT		CONSTANT COEFFICIENT
			EER (dB)	STP (dB)	CON (dB)
SET 1	21	-10.42	-121.25	-121.66	-86.98
SET 2	23	-9.98	-120.81	-116.40	-82.82
SET 3	17	-11.81	-122.37	-120.74	-86.67
SET 4	21	-10.74	-121.08	-96.21	-87.91
SET 5	20	-10.44	-120.76	-91.55	-80.73

Table1: Experimental results obtained using a varying and a constant coefficient when  $|t_n - n\Delta| < \Delta/4$ .

DATA SET	# OF ITERATION	1ST ITERATION (ESR)dB	VARYING COEFFICIENT		CONSTANT COEFFICIENT
			EER (dB)	STP (dB)	CON (dB)
SET 1	83	-5.50	-100.55	-87.16	-36.69
SET 2	236	-5.19	-100.67	-116.99	-29.20
SET 3	66	-6.55	-101.01	-98.87	-32.99
SET 4	67	-5.60	-100.37	-88.59	-28.97
SET 5	57	-5.06	-100.44	-76.49	-38.26

Table2: Experimental results obtained using a varying and a constant coefficient when  $|t_n - n\Delta| < \Delta/2$ .

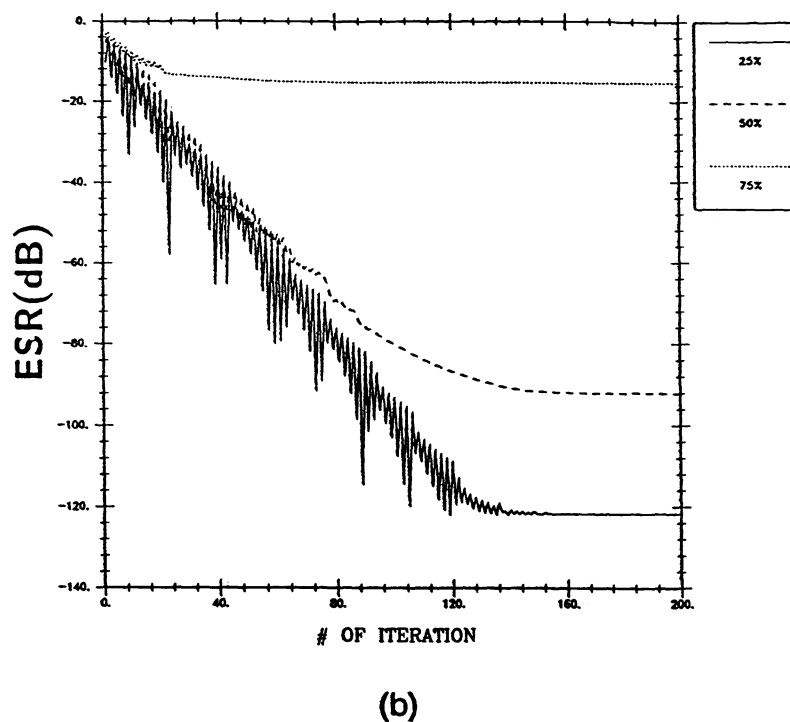
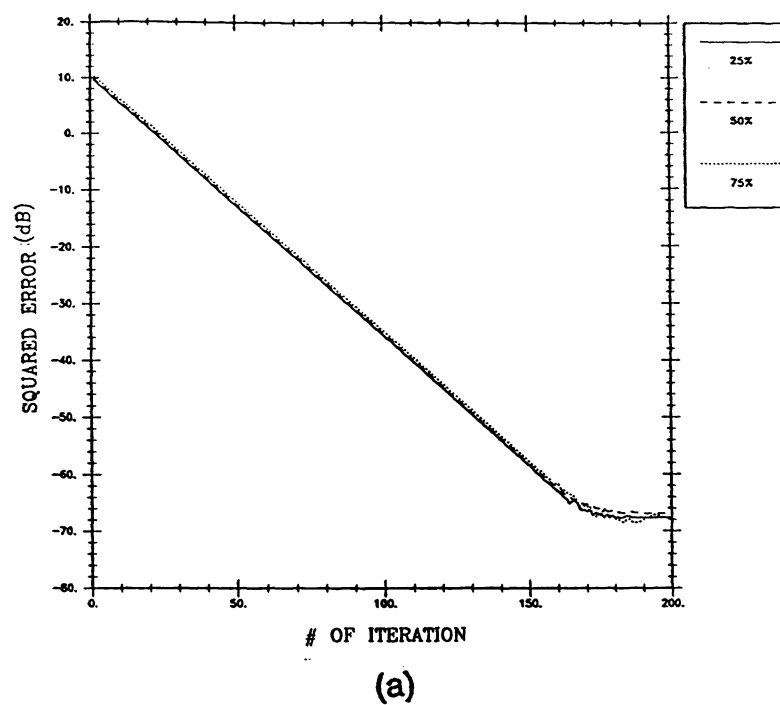


Figure 4: (a) The reduction of distance between two consecutive signal and (b) the ESR of the estimator for the signal in dB when  $|t_n - n\Delta| < \Delta/4$ ,  $\Delta/2$  and  $3\Delta/4$ .